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Trailing Vortex Filament: A Novel Similarity Analysis

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Nomenclature

$A, A1, A2$	=	constants
CDo	=	drag coefficients
D	=	drag of body producing the vortex, N/m^2
F_1, F_2, F_3	=	similarity functions
G_1, G_2, G_3	=	similarity functions
$\tilde{o}$	=	order of magnitude
P	=	static pressure, N/m2
R_e	=	flow Reynolds number $(W_\infty z/\upsilon)$
r, θ, z	=	radial, azimuthal, and axial coordinates, m
r_c	=	core radius, m/s
t	=	time, s
U, V, W_{zd}	=	nondimensional radial, tangential, and axial deficit velocity components
V_r, V_z, V_θ	=	radial, axial, and tangential velocity components, m/s
V_{zd}	=	deficit velocity, $W_\infty - V_z$, m/s
$V_{\theta c}$	=	core tangential velocity, $\Gamma_\infty/2\pi r_c$, m/s
υ	=	fluid kinematic viscosity, m^2/s
W_∞	=	freestream velocity, m/s
β	=	similarity variable, $\eta/\sqrt{\zeta}$
Γ_∞	=	vortex circulation, m^2/s
ΔP	=	nondimensional pressure, $P - P_\infty/\rho V\theta_c^2$
ε	=	very small number
ζ	=	nondimensional axial distance, z/r_c
η	=	nondimensional radial distance, r/r_c
Π_1, Π_2	=	nondimensional pressure
ρ	=	fluid density, kg/m^3
τ	=	nondimensional time, $\upsilon t/r_c^2$
ϕ	=	similarity variable, $\sqrt{(\tau + \zeta)}$
Ω_z	=	axial vorticity

Subscripts

c	=	vortex core
z	=	properties along axial direction
zd	=	refers to deficit components
∞	=	properties far away from the vortex center

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I. Introduction

IN THIS Note, a theoretical similarity analysis with regard to the time decay of a viscous trailing vortex line filament in a nonzero meridional flow is derived. A new similarity variable $[\phi = \sqrt{(\tau + \zeta)}]$ that combined the radial, axial, and time parameters is also given and was used to transform the partial differential equations into an ordinary set. The reduced form of the governing equation was found to be identical to the steady-state solution given previously by Newman [1]. Therefore, an exact solution to the time-decay problem is found by variable transformation rather than having formally to solve the time-dependent partial differential equations. Results are verified using available sets of experimental measurements. The present study concluded that, the time-axial decay phenomenon of trailing vortices attains self-similar structure behavior along both the axial and time frame of references.

The diffusion of the vorticity and decay of the velocity are both important in several applications. For instance, attention to the hazard shed by the trailing vortices produced by heavily loaded aircraft has simulated research into one of the oldest topics in the fluid mechanics which is the study of flow with concentrated vortices in free motion. Because these eddies are strong and persistent enough to cause a sort of danger to the following aircraft, it is clearly recommended to be able to predict the formation, locations, and persistence of such vortices as well as to understand the basic mechanisms by which vortex hazards will be dissipated. Many researches have been done aiming to understand the complexity of vortex flows, but only simplified algebraic models have been proposed in the past to describe this flow structure. For example, the induced swirl velocity at larger distances from the center of rotation reaches asymptotically to that of a potential (free) vortex. In addition, several other steady-state models are proposed to better approximate the tangential velocity, Rankine [2], Burgers [3], Scully [4], and Vatistas et al. [5].

The subset of viscous vortices, which are diffusing in a zero meridional flow, has been thoroughly examined in the past and many formulations exist in the open literature. For instance, Oseen [6], Hamel [7], and Lamb [8] examined the temporal decay of a potential vortex in a zero meridional field. A decaying vortex with remarkable properties was derived; see Taylor [9] by noting the similarity of vorticity diffusion to that of the heat conduction equation. Rott [10] produced yet another noteworthy time-dependent solution to the vortex equations by including the radial convective acceleration term in the tangential momentum equation. A solution in terms of a hypergeometric series was given by Kirde [11], but the solution is mathematically complex. Bellamy-Knights [12] developed a two-cell diminishing vortex where the steady vortex by Sullivan [13] is considered to be one of a subset asymptotic solution. A recent theoretical analysis concerning the time decay of laminar vortices due to the viscous effects was also developed by Aboelkassem et al [14]. In their study, a general decay formula for both Taylor and Oseen-like vortices is given in terms of Fourier–Bessel functions. Decaying of viscous vortices in a field, at which both the radial and axial velocity components exist, has not been equally explored compared to the flow with zero meridional assumptions. Vatistas et al. [15] focus on the extension of the time decay of the *n* family of vortices, but the analysis is limited to a high vortex Reynolds number. Vatistas and Aboelkassem [16] have established a general space-time analogy quality for self-similar intense vortices and have elaborated on some unique properties applicable to all simple vortex flows of this type.

The dissipation of trailing vortices along the axial directions was first given in detail by Newman [1]. The equations that govern this particular type are valid for flow at relatively moderate to high Reynolds numbers. The equations are linearized under the assumptions that both the swirl and axial deficit velocities are small compared with that in the freestream. Because the trailing vortices are strong, the axial deficit velocity appears and replaces the weak longitudinal velocity within the vortex filament. One should notice that Newman's assumptions have successfully decoupled momentum equations from each other and a closed form solution is indeed predicted. The steady, axisymmetric, and axially dependent flowfield within a line trailing vortex was modeled also in detail; see Batchelor [17]. The tangential momentum is coupled to the axial one via the pressure distribution.

On the experimental and computational side, a large-eddy simulation was carried out to study computationally the response of vortices with axial deficits to random and controlled disturbances at high Reynolds numbers; see Ragab and Sreedhar [18]. The laser doppler velocimetry measurements are obtained to elaborate on the cavitations phenomenon by tip vortices [19]. Limited experimental data are available in the open literature with regard to the temporal decay of line vortex filaments. For instance, a typical Rankine vortex of forced-free motion was produced by spinning a single thin walled cylinder inside a larger water tank [20]. Another remarkable time-decay measurement of a monopolar type of vortices within a stratified fluid is examined experimentally. Two different techniques have been used to create Taylor-like vortices. In addition, a theoretical extended decay model was derived by Trieling and van Heijst [21]. Detailed measurements to study the dynamic structure of the monopolar vortices are recently given by Flor and Eames [22]. Phillips and Graham [23] have carried out another remarkable experiment to conduct several data sets for the mean and turbulent parameters in a trailing vortex.

II. Mathematical Model

The present analysis is valid only after the roll up of the shear layer phase has been completed, that is, along the intermediate and far-field downstream domain only. In other words, there is no energy feed to the generated vortex. Therefore, due to the action viscosity, the crossflow velocity components and the associated axial vorticity start to dissipate. Consider the motion of time-dependent, incompressible, axisymmetric, laminar, intense, and isolated trailing vortex filaments. By extending the steady-state analysis given previously by Newman [1] to include the temporal effects, the time-dependent equations that govern this particular flow in dimensional form are given throughout Eqs. (1–4), where the axial velocity can be expressed as a function of the deficit velocity, that is, $V_z = W_\infty - V_{zd}$. Based on the classical order of magnitude analysis [1] as well as some experimental facts [24,25], the axial deficit, radial, and tangential velocity components induced by a single trailing vortex line filament are known to be small compared with the freestream velocity, that is, $W_\infty > V_\theta$, V_{zd}, and V_r. Moreover, the swirl velocity as well as the axial deficit velocity is of order higher than the radial and axial components, that is, V_θ and $V_{zd} > V_r$. Based on the latter, if we assume that W_∞ is $\tilde{o}$ (1) therefore, V_θ and V_{zd} are $\tilde{o}$ (ε) where (ε) is a very small number. Therefore, the equations are taking the form of

$$\frac{1}{r}\frac{\partial(rV_r)}{\partial r} - \frac{\partial V_{zd}}{\partial z} = 0 \qquad \frac{1}{\varepsilon}\frac{(\varepsilon\ \varepsilon^2)}{\varepsilon} \qquad \frac{\varepsilon}{1} \tag{1}$$

$$\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + (W_\infty - V_{zd})\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \upsilon\left\{\frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r}\frac{\partial V_r}{\partial r} + \frac{\partial^2 V_r}{\partial z^2} - \frac{V_r}{r^2}\right\} \tag{2}$$

$$\frac{\varepsilon^2}{1} \quad \varepsilon\frac{\varepsilon^2}{\varepsilon} \quad (1 \quad \varepsilon)\frac{\varepsilon^2}{1} \quad \frac{\varepsilon^2}{\varepsilon} = \quad \frac{\varepsilon^2}{\varepsilon} \quad \varepsilon^2\left\{\frac{\varepsilon}{\varepsilon^2} \quad \frac{1}{\varepsilon}\frac{\varepsilon^2}{\varepsilon} \quad \frac{\varepsilon^2}{1} \quad \frac{\varepsilon^2}{\varepsilon^2}\right\}$$

$$\frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + (W_\infty - V_{zd})\frac{\partial V_\theta}{\partial r} - \frac{V_r V_\theta}{r} = \upsilon\left\{\frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial V_\theta}{\partial r} + \frac{\partial^2 V_\theta}{\partial z^2} - \frac{V_\theta}{r^2}\right\}$$

$$\frac{\varepsilon}{1} \quad \varepsilon^2\frac{\varepsilon}{\varepsilon} \quad (1 \quad \varepsilon)\frac{\varepsilon}{\varepsilon} \quad \frac{\varepsilon^2\varepsilon}{\varepsilon} = \varepsilon^2\left\{\frac{\varepsilon}{\varepsilon^2} \quad \frac{1}{\varepsilon}\frac{\varepsilon}{\varepsilon} \quad \frac{\varepsilon}{1} \quad \frac{\varepsilon}{\varepsilon^2}\right\} \tag{3}$$

$$\frac{\partial V_{zd}}{\partial t} + V_r\frac{\partial V_{zd}}{\partial r} + (W_\infty - V_{zd})\frac{\partial V_{zd}}{\partial z} = \frac{1}{\rho}\frac{\partial P}{\partial z} + \upsilon\left\{\frac{\partial^2 V_{zd}}{\partial r^2} + \frac{1}{r}\frac{\partial V_{zd}}{\partial r} + \frac{\partial^2 V_{zd}}{\partial z^2}\right\}$$

$$\frac{\varepsilon}{1} \quad \varepsilon^2\frac{\varepsilon}{\varepsilon} \quad (1 \quad \varepsilon)\frac{\varepsilon}{1} \quad = \frac{\varepsilon^2}{1} \quad \varepsilon^2\left\{\frac{\varepsilon}{\varepsilon^2} \quad \frac{1}{\varepsilon}\frac{\varepsilon}{\varepsilon} \quad \frac{\varepsilon}{1}\right\} \tag{4}$$

Neglecting all terms of (ε^2) and higher, therefore the governing equations simplify into the following set in a nondimensional form:

$$\begin{aligned}
&\frac{1}{\eta}\frac{\partial(\eta U)}{\partial \eta} - \frac{\partial W_{zd}}{\partial \zeta} = 0 \qquad \frac{V^2}{\eta} = \frac{\partial \Delta P}{\partial \eta}\\
&\frac{\partial V}{\partial \tau} + (R_e\bar{W}_\infty)\frac{\partial V}{\partial \zeta} = \left\{\frac{\partial^2 V}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial V}{\partial \eta} - \frac{V}{\eta^2}\right\}\\
&\frac{\partial W_{zd}}{\partial \tau} + (R_e\bar{W}_\infty)\frac{\partial W_{zd}}{\partial \zeta} = \left\{\frac{\partial^2 W_{zd}}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial W_{zd}}{\partial \eta}\right\}
\end{aligned} \tag{5}$$

where

$$\eta = \frac{r}{r_c}, \qquad \zeta = \frac{z}{r_c}, \qquad \tau = \frac{\upsilon t}{r_c^2} \qquad R_e = \frac{V_{\theta c} r_c}{\upsilon}$$

$$\Delta P = \frac{p - p_\infty}{\rho V_{\theta c}^2}, \qquad V_{\theta c} = \frac{\Gamma_\infty}{2\pi r_c} \qquad U = \frac{V_r}{V_{\theta c}}, \qquad V = \frac{V_\theta}{V_{\theta c}}$$

$$W_{zd} = \frac{V_{zd}}{V_{\theta c}}, \qquad \bar{W}_\infty = \frac{W_\infty}{V_{\theta c}}$$

The required initial and boundary conditions for this particular formulation are as follows:

1) $\tau = \tau_{\text{in}}$, $U(\tau_{\text{in}}, \eta, \zeta) = f_1$, $V(\tau_{\text{in}}, \eta) = f_2$, and $W_{zd}(\tau_{\text{in}}, \eta, \zeta) = f_3$;
2) $\eta = 0$, $V = U = 0$, and $\partial W_{zd}/\partial\eta = 0$;
3) $\eta \to \infty$, $\eta V = 1$, and $U = W_{zd} = 0$;
4) $\zeta \to \infty$, and $V = U = W_{zd} = 0$.

Clearly, the above system of equations is described in a partial differential form. In addition, the radial momentum suggests that the static pressure develops to balance the centrifugal forces and the tangential momentum is completely decoupled from the axial momentum equation. Consequently, the axial velocity is related to radial velocity via the mass conservation alone. An analytical solution to the steady-state subset equations is well documented in the literature [1,17]. For instance, the steady-state solution by Newman [1] is presented as a benchmark case study in order to verify the similarity hypothesis. The subset equations that govern the axial decay of a steady-viscous vortex's filament are given directly using Eqs. (5), by neglecting all the derivatives with respect to time. In addition, the velocity and pressure fields are assumed to depend on the axial and radial directions alone, that is, pure axisymmetric vortex, therefore,

$$\frac{1}{\eta}\frac{\partial(\eta U)}{\partial\eta}-\frac{\partial W_{zd}}{\partial\zeta}=0 \qquad \frac{V^2}{\eta}=\frac{\partial\Delta P}{\partial\eta}$$
$$(R_e\bar{W}_\infty)\frac{\partial V}{\partial\zeta}=\left\{\frac{\partial^2 V}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial V}{\partial\eta}-\frac{V}{\eta^2}\right\} \tag{6}$$
$$(R_e\bar{W}_\infty)\frac{\partial W_{zd}}{\partial\zeta}=\left\{\frac{\partial^2 W_{zd}}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial W_{zd}}{\partial\eta}\right\}$$

subject to the boundary conditions:
1) $\eta=0$, $U=V=0$, and $\partial W_{zd}/\partial\eta=0$;
2) $\eta\to\infty$, $\eta V=1$, and $U=W_{zd}=0$;
3) $\zeta\to\infty$, and $V=U=W_{zd}=0$.

The solution to the above equations is known and given by the following expressions:

$$U(\eta,\xi)=\frac{-A\eta}{2\xi^2}e^{[\frac{-R_e\bar{W}_\infty\eta^2}{4\xi}]}$$
$$V(\eta,\xi)=\frac{1}{\eta}(1-e^{[\frac{-R_e\bar{W}_\infty\eta^2}{4\xi}]}) \quad\text{and}\quad W(\eta,\xi)=\frac{A}{\xi}e^{[\frac{-R_e\bar{W}_\infty\eta^2}{4\xi}]} \tag{7}$$

The constant A can be determined by using the momentum considerations provided that the swirl momentum is decoupled from the axial-radial momenta equations.

III. Novel Similarity Variables

The similarity technique is an approach which is capable of transforming the partial differential equation into an ordinary form. Therefore, the system equations (6) are reconsidered and represented in terms of a similarity variable (β) that fuses the axial and radial coordinates into a single independent parameter. The objectives of representing the above steady equations in a similarity form are to show the similarity method applicability in a simple case where the solution is well knowm [1] and also to establish a base for further complex cases such as the time-dependent situation. Generally, the similarity approach suggests that a solution might exist to the modified-reduced order equation if the objective flowfield functions have symmetry properties and/or they differ at two locations by only a scaling factor. Based on the dimensional analysis and the self-similarity hypostasis besides assuming that the vortex filament decays under the action of viscosity alone are used to relate the swirl velocity to some other physical parameters involved in the decay process. Therefore, the nondimensional swirl velocity may be expressed mathematically in the following form:

$$V(\eta,\zeta,\tau)=F(\zeta,\tau)G\left[\frac{\eta}{H(\zeta,\tau)}\right] \tag{8}$$

where the geometrical coordinates are designated by η, ζ and the time is τ. The function (F) determines the scale of the field function of interest (V), such that (G) is an order-one function which varies solely with the similarity variable $\phi=\eta/H(\zeta,t)$. For details, see Lugt [26].

In view of the latter quality, the following steady-state similarity variables are proposed. These function groups vary solely with a single parameter (β), which combined the axial and radial coordinates. Hence, the transformation relations are given as follows:

$$\beta=\frac{\eta}{\sqrt{\zeta}},\qquad U(\eta,\zeta)=\frac{G_1(\beta)}{\zeta\sqrt{\zeta}},\qquad V(\eta,\zeta)=\frac{G_2(\beta)}{\sqrt{\zeta}}$$
$$W_{zd}(\eta,\zeta)=\frac{G_3(\beta)}{\zeta},\qquad \Delta P(\eta,\zeta)=\frac{\Pi_1(\beta)}{\zeta},\qquad A_1=R_e\bar{W}_\infty \tag{9}$$

It is mathematically straightforward to show that the governing equations (6) can be reduced to an ordinary set by making use of the above similarity variables given by Eqs. (9), yielding

$$\frac{1}{\beta}\frac{d(\beta G_1)}{d\beta}+\frac{1}{2}\left(\beta\frac{dG_3}{d\beta}+2G_3\right)=0 \qquad \frac{G_2^2}{\beta}=\frac{d\Pi_1}{d\beta}$$
$$\frac{d^2G_2}{d\beta^2}+\left(\frac{1}{\beta}+\frac{\beta}{2}A_1\right)\frac{dG_2}{d\beta}+\left(\frac{1}{2}A_1-\frac{1}{\beta^2}\right)G_2=0 \tag{10}$$
$$\frac{d^2G_3}{d\beta^2}+\left(\frac{1}{\beta}+\frac{\beta}{2}A_1\right)\frac{dG_3}{d\beta}+(A_1)G_3=0$$

subject to the following boundary conditions:
1) $\beta=0$, $G_1=G_2=0$, and $dG_3/d\beta=0$;
2) $\beta\to\infty$, $\beta G_2=1$, and $G_1=G_3=0$.

An exact solution to the system of Eqs. (10) subjected to the above spatial boundary conditions is easy to obtain using the classical methods of integration. However due to the fact that the two systems of Eqs. (6) and (10) and their boundary conditions are identical, the solutions to Eqs. (10) are given directly throughout a variable transformation rather than physically solving the ordinary equations (10).

$$G_1(\beta)=\frac{-A\beta}{2}e^{(\frac{-R_e\bar{W}_\infty\beta^2}{4})}$$
$$G_2(\beta)=\frac{1}{\beta}(1-e^{(\frac{-R_e\bar{W}_\infty\beta^2}{4})}) \quad\text{and}\quad G_3(\beta)=Ae^{(\frac{-R_e\bar{W}_\infty\beta^2}{4})} \tag{11}$$

The above solution is a function only in a single parameter (β), which assists the vortex self-similarity structure and it is indeed satisfying the original partial differential equations (6). The main conclusion from this part is that the similarity analysis may be used to solve more complex situations such as the time-dependent vortex flow, that is, the time decay of vortices.

A new similarity variable (ϕ) is introduced to combine the radial, axial, and time parameters into a single variable. Consequently, the partial differential equations (5) that govern this particular flow are transformed into an ordinary set. Based on dimensional analysis and analogy hypothesis, we propose the following time-dependent similarity groups, where

$$\phi=\frac{\eta}{\sqrt{\tau+\zeta}},\qquad U(\eta,\zeta,\tau)=\frac{F_1(\phi)}{(\tau+\zeta)\sqrt{\tau+\zeta}}$$
$$V(\eta,\zeta,\tau)=\frac{F_2(\phi)}{\sqrt{\tau+\zeta}}\qquad W_{zd}(\eta,\zeta,\tau)=\frac{\Pi_2(\phi)}{(\tau+\zeta)} \tag{12}$$
$$A_2=1+R_e\bar{W}_\infty$$

Using the standard chain of variable differentiation it is easy to show that the governing equations and the associated boundary conditions that describes the time decay of a trailing vortex's filament, Eqs. (5) are reduced to a simpler ordinary differential form:

$$\frac{1}{\phi}\frac{d(\phi F_1)}{d\phi}+\frac{1}{2}\left(\phi\frac{dF_3}{d\phi}+2F_3\right)=0 \qquad \frac{F_2^2}{\phi}=\frac{d\Pi_2}{d\phi}$$
$$\frac{d^2F_2}{d\phi^2}+\left(\frac{1}{\phi}+\frac{\phi}{2}A_2\right)\frac{dF_2}{d\phi}+\left(\frac{1}{2}A_2-\frac{1}{\phi^2}\right)F_2=0 \tag{13}$$
$$\frac{d^2F_3}{d\phi^2}+\left(\frac{1}{\phi}+\frac{\phi}{2}A_2\right)\frac{dF_3}{d\phi}+(A_2)F_3=0$$

subject to the following boundary conditions:
1) $\phi=0$, $F_1=F_2=0$, and $dF_3/d\phi=0$;
2) $\phi\to\infty$, $\phi F_2=1$, and $F_1=F_3=0$.

Equations (10) and (13) as well as their boundary conditions are exactly the same, since the solution to the steady-state subset equations exists; therefore, the solution to the time-decay problem, that is, Eq. (13) is then given through a variables transformation rather than formally solving the partial differential equations, yielding

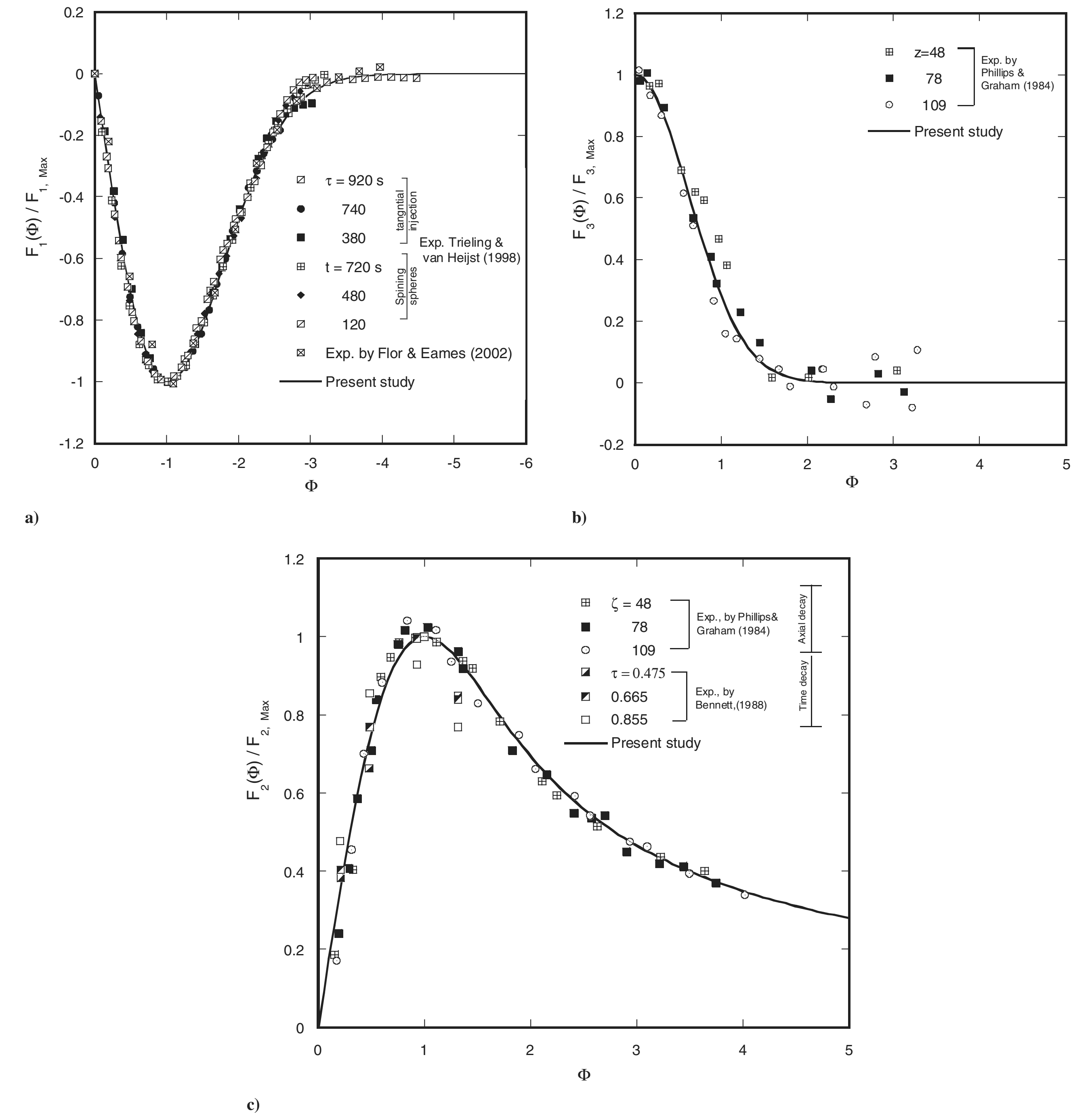

Fig. 1 Time-space similarity solution; a)–c) radial, axial, and swirl velocity components.

$$F_1(\phi) = \frac{-A\phi}{2} e^{\left(\frac{-(1+R_e\bar{W}_\infty)\phi^2}{4}\right)}$$
$$F_2(\phi) = \frac{1}{\phi}\left(1 - e^{\left(\frac{-(1+R_e\bar{W}_\infty)\phi^2}{4}\right)}\right) \quad \text{and} \quad F_3(\phi) = A e^{\left(\frac{-(1+R_e\bar{W}_\infty)\phi^2}{4}\right)} \tag{14}$$

The above expressions vary only with a single parameter (ϕ). The corresponding velocity components as a function of space and time variables are

$$U(\eta,\zeta,\tau) = -\frac{A\eta}{2(\tau+\zeta)^2}\exp\left(-\frac{(1+R_e\bar{W}_\infty)\eta^2}{4(\tau+\zeta)}\right)$$
$$V(\eta,\zeta,\tau) = \frac{1}{\eta}\left[1 - \exp\left(-\frac{(1+R_e\bar{W}_\infty)\eta^2}{4(\tau+\zeta)}\right)\right] \tag{15}$$
$$W_{zd}(\eta,\zeta,\tau) = \frac{A}{(\tau+\zeta)}\exp\left(-\frac{(1+R_e\bar{W}_\infty)\eta^2}{4(\tau+\zeta)}\right)$$

Similarly, an expression that describes the time decay of a vorticity as a function of both time and spatial locations is given as follows:

$$\Omega_z = \frac{1}{\eta}\frac{\partial(\eta V)}{\partial\eta} \quad \text{or} \quad \Omega_z = \frac{1+R_e\bar{W}_\infty}{2}\exp\left(\frac{(1+R_e\bar{W}_\infty)}{4(\tau+\zeta)}\eta^2\right) \tag{16}$$

The present similarity analysis has shown that the tangential momentum is decoupled from the axial one; therefore both radial and axial velocity are not affected by the tangential velocity distribution, which is not the case in realistic line vortices. However, it provides some information about the time-decay problem. The axial momentum gives an explicit formula for the axial velocity distribution. The radial velocity is directly calculated from the continuity equation. The pressure force is balanced by the centrifugal forces; see the radial momentum equation. The present model solution is therefore the offspring of a decaying vortex, originally potential vortex [6,7] with singular radial and axial velocities. The swirl velocity profile has the traditional Oseen-like shape, the axial velocity has a jetlike distribution, and the radial exhibits a monopolarlike (Taylor's) profile. Renormalization of the velocity components and radial coordinates by the maximum values and all of the axial-time decay profiles should fall into a single curve, self-similar structure. Based on the proposed similarity analysis, the decay phenomenon is the same if it is viewed from a moving frame of reference and being steady or from the stationary frame of reference and being time dependent. The present results are verified using different axial as well as time-decay laboratory measurements for different type of vortices, namely, Oseen's and Taylor's, Figs. 1a–1c.

In this similarity study, we have shown that steady vortices can be transformed into time decaying and vice versa, and this transformation depends solely on the similarity variable $[\phi = \sqrt{(\tau + \zeta)}]$. In other words, if ϕ is indeed the variable that performs this transformation then both, steady and decaying vortices, irrespective of their size, strength, and time level, should collapse into a single curve. The observed tangential velocities from various experiments are plotted in Figs. 1a–1c. The plot shows that all the data collapsed reasonably into the proposed theoretical profile. It is indeed remarkable that such a correlation was attained in spite of the difficulties encountered with the experimental technique which are mainly associated with the accurate location of the vortex center from the velocity measurements and/or the unsteady character of the vortex core due to the existence of various forms of instability waves.

Additionally, although in these experiments the tangential velocity is in fact larger than the other two components [23], it may not necessarily be several orders of magnitude greater. The latter suggests that in reality, the analogy might be preserved under less restrictive conditions than those imposed by the present simple theory. This particular character of the azimuthal flow properties like the tangential velocity, vorticity, circulation, and pressure is also retained by vortices with substantially different meridional flow such as the trailing vortices in the Newman [1] and Batchelor [17] flow regimes.

IV. Conclusions

The classical spatial-decay analysis of a trailing vortex's line filament is extended to include the temporal effects. Based on the order of magnitude argument, the time-dependent governing equations are reduced into a simpler form. New similarity variables are introduced and have been used to transform the time-dependent partial differential equations into an ordinary set. The latter have proven to be identical to the steady analysis given previously by Newman [1]. Consequently, the solution to the time-decay problem is found by analogy to the steady-state subset and without having formally to solve the governing equations. In other words, the time-decay problem is similar to the axial decay one. In addition, the present theoretical study shows the space-time analogy shared by a group of self-similar structure vortices. The steady vortices can be transformed into time decaying and vice versa. This transformation depends solely on the similarity variable (ϕ). The present results are confirmed by available experimental measurements and have shown that steady and decaying vortices, irrespective of their size, strength, and time level, should collapse into a single curve, that is, the space-time self-similarity hypothesis is indeed validated.

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